1. Use the definition of the definite integral to evaluate
\[ \int_{-1}^{3} 2x^2 \, dx. \]

2. Apply the comparison test for improper integrals to decide the convergence or divergence of each of the following:
\[ \int_{1}^{\infty} \frac{1}{1 + x^4} \, dx, \quad \int_{1}^{\infty} \frac{1}{\sqrt[3]{1 + x^3}} \, dx, \quad \int_{1}^{\infty} \sqrt{\frac{e^{-x}}{x^2}} \, dx, \quad \int_{1}^{\infty} \frac{\sin^4 x}{x^2} \, dx. \]

3. Consider the circle \( C \) with center \((0, R)\) and radius \(r\), where \(r < R\).
   
   (a) Find the Cartesian equation of this circle;
   
   (b) Revolving \( C \) around the \(x\)-axis generates a torus. Find the volume of this torus in terms of \(r\) and \(R\). You may want to know that \(\int_{-a}^{a} \sqrt{a^2 - u^2} \, du = \frac{\pi a^3}{2}\).
   
   (c) Revolving just the upper half of \( C \) about the \(z\)-axis generates the “outer half” of the surface of the torus. Revolving the lower half of \( C \) generate the “inner half” of the torus. Find the volume of these two “halves” of the torus. Should they be equal? Are they?
   
   **NOTE:** It should take only a small modification of your work from part (b) to compute these areas.

4. A solid is generated by rotating the region bounded by the \(x\)-axis, the \(y\)-axis and the curve \(y = f(x)\) around the \(x\)-axis. Here, \(f\) is a positive function and \(x \geq 0\). The volume generated by the part of the curve from \(x = 0\) to \(x = b\) is \(b^2\) for all \(b > 0\). Find the function \(f\). **Hint:** Set up the integral, and think back to the FTC.

5. If \(y = f(x)\) is a nice (ie, differentiable) curve which is revolved around the \(x\)-axis, we can compute the surface area of the resulting object (for example, the surface area of an apple is its peel) by computing \(\int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx\). This should look somewhat familiar: the square root term is the arc length of the curve \(y = f(x)\) from \(a\) to \(b\), and \(2\pi f(x)\) should make you think about a circle (why?). Write an essay that explains why this integral gives the surface area. Be sure to draw some pictures to help explain your ideas.

6. Consider a hemispherical loaf of bread of radius \(r\), with all of the crust on the flat horizontal bottom trimmed off. Slice the loaf along parallel vertical planes into 5 pieces of equal width.

   (a) Intuitively, which slice do you think has the most crust? Why?
   
   (b) Calculate the area of the crust on each slice and find out which slice really has the most crust. (NOTE: See previous problem.)
   
   (c) Suppose you slice the loaf into \(n\) pieces of equal width instead of 5. Now which slice has the most crust?

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.
\]
7. Gabriel's Horn is obtained by revolving the graph of \( f(x) = \frac{1}{x} \), on the interval \([1, \infty)\), about the \( x \)-axis. Show that the volume of horn is finite, but the surface area is not; i.e. the inside of the horn could be filled with a finite amount of paint, but the outside of the horn could not be painted by a finite amount of paint.

8. Consider the curve \( y = 8x - 27x^3 \). Let \( c > 0 \) be a constant such that the horizontal line \( y = c \) intersects the curve in two places in the first quadrant. Find the value of \( c \) such that the area in the first quadrant below \( y = c \) and above \( y = 8x - 27x^3 \) and bounded by the \( y \)-axis is equal to the area in the first quadrant which is above \( y = c \) and below \( y = 8x - 27x^3 \).

9. It takes a force of 21,714 lb to compress a coil spring assembly on a New York City subway car from its free height of 8 inches to its fully compressed height of 5 inches. How much work does it take to compress the assembly the first half an inch? The second half an inch?

10. A tank full of water has the shape of a paraboloid of revolution; that is, the shape is obtained by revolving a parabola about the \( y \) axis. If the height \( H \) is 4 ft and the radius at the top is \( R = 4 \) ft, find the work required to pump the water out the top of the tank. (water weighs 62.5 lb/ft\(^3\)) If the pump breaks after 2000 ft-lbs of work has been done, what is the height of the water left in the tank?

11. Suppose that \( C > 0 \). Given a starting value \( a_1 \), define the sequence \( \{a_n\} \) recursively as follows:
\[
a_{n+1} = \frac{1}{2}(a_n + \frac{C}{a_n}), \quad n \geq 1.
\]
Prove that if \( L = \lim_{n \to \infty} a_n \) exists then \( L = \frac{1}{2}\sqrt{C} \).

12. For the sequence \( a_n = (1 + \frac{1}{n})^4 \), decide whether it converges or diverges. If it converges, find the limit.

13. The Sierpinski Triangle is constructed in the following way: Start with an equilateral triangle of side length \( s \) and area \( A \). Remove the equilateral triangle which has its vertices at the midpoint of each of the original sides from the center of the original triangle. What remains is a collection of three triangles, each with side length \( s/2 \) and area \( A/4 \). Perform the center-removing operation on each of these three triangles. The result will be 9 triangles of side length \( s/4 \). Remove the centers of these 9 triangles, and so on.... See the figure.

(a) Use geometric series arguments to show that the total area removed is \( A \), i.e. the area of the Sierpinski triangle is 0.

(b) Show that the length of the boundary of the Sierpinski triangle is infinite.