May 24, 2004
From the problem set for exam one: 2–5, 7, 9, 11
From the problem set for exam two: 3–5, 7, 9, 11, 12

1. Use the definition of the definite integral to evaluate

\[ \int_{-1}^{1} (3x - x^2) \, dx. \]

2. The Sierpinski Triangle is constructed in the following way: Start with an equilateral triangle of side length \( s \) and area \( A \). Remove the equilateral triangle which has its vertices at the midpoint of each of the original sides from the center of the original triangle. What remains is a collection of three triangles, each with side length \( s/2 \) and area \( A/4 \). Perform the center-removing operation on each of these three triangles. The result will be 9 triangles of side length \( s/4 \). Remove the centers of these 9 triangles, and so on. . . . See the figure.

(a) Use geometric series arguments to show that the total area removed is \( A \), i.e. the area of the Sierpinski triangle is 0.
(b) Show that the length of the boundary of the Sierpinski triangle is infinite.

3. Use integration by substitution to show that \( \int f \left( \sqrt{x} \right) \, dx = \int 2u f(u) \, du \). Calculate \( \int \sin \sqrt{x} \, dx \).

4. A mathematical operation that takes a function as input and produces another function as output is sometimes referred to as a transform. An important example is the Laplace transform \( \mathcal{L} \), defined by

\[ \mathcal{L}[f(x)] = \int_{0}^{\infty} e^{-px} f(x) \, dx = F(p). \]

Notice that \( \mathcal{L} \) turns a function of \( x \) into a function of the parameter \( p \). Find the Laplace transform of the functions \( f(x) = 1 \), \( g(x) = x \) and \( h(x) = e^{5x} \).
(assume in the first two cases that \( p > 0 \) and in the third case \( p > 5 \))

5. Find the limit:

\[ \lim_{h \to 0} \frac{\int_{1}^{1+h} \sqrt{x^5 + 8} \, dx}{h}. \]

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4} \]
6. Starting with 
\[ f(x) = \sum_{n=0}^{\infty} x^n, \]
find the sum of the series:
(a) \[ \sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1 \]
(b) \[ \sum_{n=1}^{\infty} \frac{n}{2^n} \]
(c) \[ \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1 \]
(d) \[ \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \]
(e) \[ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \]

7. Use either a direct comparison or the limit comparison test to decide the convergence or divergence of each of the following:
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{1 + n^2}} \quad \sum_{n=3}^{\infty} \frac{n}{(n+3)^2} \quad \sum_{n=1}^{\infty} \frac{1}{2n^2 - n} \quad \sum_{n=1}^{\infty} \frac{5\sin^2 n}{n\sqrt{n}} \]

8. When a plane region \( R \) is revolved around an axis \( A \), a solid of revolution \( S \) is generated. Let \( CM \) be the center of mass of \( R \), and \( \bar{C} \) be the circle generated when \( CM \) is revolved around the axis \( A \). The Theorem of Pappus says that 
\[ \text{Volume of } S = (\text{Circumference of } \bar{C}) \cdot (\text{Area of } R). \]

Suppose \( R \) is the region between the graph of \( y = f(x) \) and the \( x \)-axis from \( x = a \) to \( x = b \), where \( 0 < a < b \) and \( f(x) > 0 \). Show that the Theorem of Pappus is true for the solid of revolution \( S \) generated by revolving \( R \) about the \( y \)-axis. [HINT Use the method of shells.]

Recall the torus from problem 10 of the second exam problem set. Use the Theorem of Pappus to find its volume. How does this answer compare with your result using washers or shells?
9. Consider a thin, homogeneous, rigid rod of length \( L \) cm and density \( d \) g/cm, aligned on the \( x \)-axis with one end fixed at the origin.

Now suppose the rod is rotating around the \( y \)-axis with angular velocity \( \omega \) rev/sec, like a helicopter rotor. Set up and calculate the integral to find the total kinetic energy of the rotating rod.

**THE PROCEDURE:** Go back to basic principles. Dissect the rod into small pieces, figure out the kinetic energy of each small bit, and then add all the contributions to the total kinetic energy with an appropriate integral. The kinetic energy of a point mass \( m \) is \( \frac{1}{2}mv^2 \) where \( v \) is the linear velocity, measured in units of cm/sec. The linear velocity of a point on the rotating rod \( r \) units away from the origin is \( 2\pi r\omega \).

10. Evaluate the definite integral

\[
\int_{0}^{1} f(x) \, dx
\]

where \( f \) is the function whose graph is shown below

![Graph of f(x)](image)

11. Here is a “proof” that 1=0. Your job is to find the bug.

On the one hand,

\[
\frac{2}{\pi} \int_{0}^{\pi} \cos^2 \theta \, d\theta = \frac{2}{\pi} \int_{0}^{\pi} \frac{1 + \cos 2\theta}{2} \, d\theta
\]

\[
= \frac{2}{\pi} \left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_{0}^{\pi}
\]

\[
= \frac{2}{\pi} \left( \frac{\pi}{2} \right) = 1.
\]

On the other hand, let \( u = \sin \theta \). Then \( \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - u^2} \) and \( du = \cos \theta \, d\theta \), so

\[
\frac{2}{\pi} \int_{0}^{\pi} \cos^2 \theta \, d\theta = \frac{2}{\pi} \int_{0}^{\pi} \cos \theta (\cos \theta) \, d\theta
\]

\[
= \frac{2}{\pi} \int_{0}^{\sin \pi} \sqrt{1 - u^2} \, du = 0.
\]
12. (a) Use Taylor’s Theorem to find the Maclaurin series for \( f(x) = e^x \).

(b) Use part (a) to find the Maclaurin series for \( e^{-x^3} \). What is the interval of convergence of this series? For what \( x \) does it converge absolutely?

(c) Calculate
\[
\int_0^2 e^{-x^3} \, dx
\]

to 4-decimal place accuracy. Explain how you know that your answer is sufficiently accurate.

13. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the book beneath it: the top book extends half its length beyond the second book, the second book extends a quarter of its length beyond the third, the third extends one sixth of its length beyond the fourth, and so on.

(a) Show, by considering the center of mass of the stack, that it is possible to stack the books so that the top book extends entirely beyond the table. Work from the top down. First find the center of mass of two books. Then use this result to find the center of mass when a third book has been added to the bottom of the stack. How many books does it take before the top book extends completely beyond the edge of the table?

(b) By considering the harmonic series, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough.