May 29, 2007

Be sure to review both previous sample exams!

1. The Sierpinski Triangle is constructed in the following way: Start with an equilateral triangle of side length $s$ and area $A$. Remove the equilateral triangle which has its vertices at the midpoint of each of the original sides from the center of the original triangle. What remains is a collection of three triangles, each with side length $s/2$ and area $A/4$. Perform the center-removing operation on each of these three triangles. The result will be 9 triangles of side length $s/4$. Remove the centers of these 9 triangles, and so on. . . . See the figure.

(a) Use geometric series arguments to show that the total area removed is $A$, i.e. the area of the Sierpinski triangle is 0.

(b) Show that the length of the boundary of the Sierpinski triangle is infinite.

2. Use integration by substitution to show that \( \int f(\sqrt{x}) \, dx = \int 2uf(u) \, du. \) Calculate \( \int \sin \sqrt{x} \, dx. \)

3. A mathematical operation that takes a function as input and produces another function as output is sometimes referred to as a transform. An important example is the Laplace transform \( \mathcal{L} \), defined by

\[
\mathcal{L}[f(x)] = \int_{0}^{\infty} e^{-px} f(x) \, dx = F(p).
\]

Notice that \( \mathcal{L} \) turns a function of \( x \) into a function of the parameter \( p \). Find the Laplace transform of the functions \( f(x) = 1, \ g(x) = x \) and \( h(x) = e^{5x}. \)

(assume in the first two cases that \( p > 0 \) and in the third case \( p > 5 \))

4. Find the limit:

\[
\lim_{h \to 0} \frac{\int_{1}^{1+h} \sqrt{x^3 + 8} \, dx}{h}.
\]

5. Starting with

\[
f(x) = \sum_{n=0}^{\infty} x^n,
\]

find the sum of the series:
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(a) \[ \sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1 \]

(b) \[ \sum_{n=1}^{\infty} \frac{n}{2^n} \]

(c) \[ \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1 \]

(d) \[ \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \]

(e) \[ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \]

6. **What can the comparison test tell you?**

   Consider the series
   \[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \] (1)

   (a) Show that this series converges by comparison with \( \sum_{k=1}^{\infty} \frac{1}{k^2} \)

   (b) It is known that \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6 \). Use partial sums to find the sum of the series given in (1).

7. Use the Taylor formula to derive the power series representation of \( f(x) = \cos 2x \) about zero. Determine the radius of convergence of the power series.

8. Find the sums of each of the following series, by finding a power series in \( x \) for which a specific value of \( x \) yields the given series.

   \[ \sum_{k=1}^{\infty} \frac{k}{2^k} \quad \sum_{k=1}^{\infty} \frac{1}{k2^k} \quad \sum_{k=1}^{\infty} \frac{2^k}{(k+1)!} \]

9. Evaluate the definite integral

   \[ \int_{0}^{1} f(x) \, dx \]
10. (a) Use Taylor’s Theorem to find the Maclaurin series for \( f(x) = e^x \).

(b) Use part (a) to find the Maclaurin series for \( e^{-x^3} \). What is the interval of convergence of this series? For what \( x \) does it converge absolutely?

(c) Calculate
\[
\int_0^{0.2} e^{-x^3} \, dx
\]
to 4-decimal place accuracy, and explain how you know that your answer has the desired accuracy.

11. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the book beneath it: the top book extends half its length beyond the second book, the second book extends a quarter of its length beyond the third, the third extends one sixth of its length beyond the fourth, and so on.

(a) Show, by considering the center of mass of the stack, that it is possible to stack the books so that the top book extends entirely beyond the table. Work from the top down. First find the center of mass of two books. Then use this result to find the center of mass when a third book has been added to the bottom of the stack. How many books does it take before the top book extends completely beyond the edge of the table?

(b) By considering the harmonic series, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough.

12. For each of the following functions, find its Taylor series about \( x = 0 \) without computing \( f^{(k)}(0) \). Instead, modify a familiar Taylor series by integrating it, differentiating it, making a substitution in it, or some combination of these.

\[
\frac{1}{(1 - x)^2}, \quad e^{-x^2}, \quad \frac{\ln(1 + x)}{x}
\]
13. Show that
\[ \int_0^1 e^x \, dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!}. \]

14. Short, complete answer.

(a) Is it true that if \( \lim_{n \to \infty} a_n = 0 \) then \( \sum_{n=1}^{\infty} a_n \) converges?
(b) Is it true that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \)?
(c) Is it true that if \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges?
(d) Is it true that if \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges?
(e) Is it true that if \( \sum c_n 6^n \) is convergent, then \( \sum c_n (-2)^n \) is also convergent?
(f) Is it true that if \( \sum c_n 6^n \) is convergent, then \( \sum c_n (7)^n \) is divergent?
(g) If \( f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots \) converges for all \( x \), what is \( f''(0) \)?
(h) Only one of the following functions is equal to its Maclaurin series on the interval \((-2, 2)\). Use Taylor’s theorem to explain which is which.

\[ f(x) = \frac{1}{1-x} \quad g(x) = \frac{1}{5-x} \]