1. Use the definition of the definite integral to evaluate
\[ \int_{-1}^{3} 2x^2 \, dx. \]

2. The gamma function \( \Gamma(x) \) is defined for all \( x > 0 \) by
\[ \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} \, dt. \]

(a) Evaluate \( \Gamma(1) \).

(b) for \( x > 1 \), show that \( \Gamma(x) = (x - 1)\Gamma(x - 1) \).
\text{Hint: use integration by parts with } u = t^{x-1}.

(c) Use parts (a) and (b) to find \( \Gamma(2) \), \( \Gamma(3) \), \( \Gamma(4) \), \( \Gamma(5) \). What is the pattern?

3. Apply the comparison test for improper integrals to decide the convergence or divergence of each of the following:
\[ \int_{1}^{\infty} \frac{1}{1 + x^4} \, dx \quad \int_{1}^{\infty} \frac{1}{\sqrt{1 + x^3}} \, dx \quad \int_{1}^{\infty} \sqrt{\frac{1 + e^{-x}}{x^2}} \, dx \quad \int_{1}^{\infty} \frac{\sin^4 x}{x^2} \, dx \]

4. \textbf{Gabriel's Horn} is obtained by revolving the graph of \( f(x) = \frac{1}{x} \), on the interval \([1, \infty)\), about the \( x \)-axis. Show that the volume of horn is finite, but the surface area is not; i.e. the inside of the horn could be filled with a finite amount of paint, but the outside of the horn could not be painted by a finite amount of paint.

5. Consider the function \( f(x) = x^2 \) on the interval \([0, 3]\), and the region \( R \) below the graph of \( f \) on this interval. Find:

(a) The average value\(^2\) \( f_{\text{ave}} \) on \([0, 3]\) and the number \( x_{\text{ave}} \) in \([0, 3]\) at which it occurs (i.e., \( f(x_{\text{ave}}) = f_{\text{ave}} \)).

(b) \( \bar{x} \), the \( x \)-coordinate of the centroid of \( R \).

(c) the value of \( x_{\text{split}} \) in \([0, 3]\) for which the vertical line \( x = x_{\text{split}} \) divides \( R \) into two regions of equal area.

Why are the numbers \( x_{\text{ave}}, x_{\text{split}} \) and \( \bar{x} \) not equal?

6. Consider the region bounded above by the curve \( y = x^{1/n} \) and below by the \( x \)-axis for \( 0 \leq x \leq 1 \).

(a) Sketch some typical regions of this type, say for \( n = 2 \) and \( n = 3 \).

(b) Find formulas (which will depend on \( n \)) for the area of this region.

(c) What happens to the area as \( n \to \infty \)? Explain why this is physically reasonable.

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4} \]

\(^1\text{The average value of a function } f \text{ over the interval } [a, b] \text{ is defined as } f_{\text{ave}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx\)
7. Consider a hemispherical loaf of bread of radius $r$, with all of the crust on the flat horizontal bottom trimmed off. Slice the loaf along parallel vertical planes into 5 pieces of equal width.

(a) Intuitively, which slice do you think has the most crust? Why?
(b) Calculate the area of the crust on each slice and find out which slice really has the most crust.
(c) Suppose you slice the loaf into $n$ pieces of equal width instead of 5. Now which slice has the most crust?

8. A tank full of water has the shape of a paraboloid of revolution; that is, the shape is obtained by revolving a parabola about the $y$ axis. If the height $H$ is 4 ft and the radius at the top is $R = 4$ ft, find the work required to pump the water out the top of the tank.

(water weighs 62.5 lb/ft$^3$)

9. What is wrong with the following calculation

$$
\int_0^3 \frac{dx}{x-1} = \ln |x-1|^3 \bigg|_0^3 = \ln 2?
$$

Evaluate the integral correctly.

10. Consider the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$, and the region bounded above by $f$ and below by the $x$-axis, over the same interval. Set up the integrals for:

(a) The volume of revolution obtained by revolving the region about the $y$-axis.
(b) The volume of revolution obtained by revolving the region about the $x$-axis.
(c) The arclength of the graph of $f$ from $(1, 1)$ to $(4, 2)$.
(d) The area of the surface of revolution obtained by revolving the graph of $f$ about the $x$-axis.

11. An elevator weighing 10,000 pounds is lifted 100 feet by winding its cable onto a winch. Neglecting friction, how much work is done if the cable weighs 10 pounds per foot?

12. The Sierpinski Triangle is constructed in the following way: Start with an equilateral triangle of side length $s$ and area $A$. Remove the equilateral triangle which has its vertices at the midpoint of each of the original sides from the center of the original triangle. What remains is a collection of three triangles, each with side length $s/2$ and area $A/4$. Perform the center-removing operation on each of these three triangles. The result will be 9 triangles of side length $s/4$. Remove the centers of these 9 triangles, and so on. . . . See the figure.

(a) Use geometric series arguments to show that the total area removed is $A$, i.e. the area of the Sierpinski triangle is 0.
(b) Show that the length of the boundary of the Sierpinski triangle is infinite.

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3The surface area is given by

$$
\int_a^b 2\pi y \, ds = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
$$

Try to justify this formula with a picture.